Extended Abstract

Thermodynamic Geometry of Ideal gas with Bose-Dunkl statistics

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ABSTRACT

In this paper, we investigate the thermodynamic geometry of an ideal gas composed of particles obeying the generalized Bose-Dunkl statistics. We first derive the generalized commutation relations for bosonic and fermionic systems with specific parity features using the Dunkl operator. Then, we compute the thermodynamic curvature of these systems and compare it with the ideal Bose gas. The results show that the thermodynamic curvature remains positive for negative values of the Dunkl parameter, similar to the Bose ideal gas, while for positive values of the parameter, the curvature initially becomes negative and then turns positive for values beyond a certain threshold. These results are consistent with the fermionic and bosonic behaviors in these systems. Additionally, the thermodynamic curvature diverges at the fugacity value of z=1 for all values of the Dunkl parameter, similar to the ideal Bose gas.

Keywords: Dunkl operator; Generalized statistics; Thermodynamic Geometry

1. INTRODUCTION

The Dunkl operator, introduced by Charles Dunkl in the late 1980s, generalizes the standard exponential function by incorporating reflection symmetries. Unlike ordinary derivatives, the Dunkl operator includes a reflection term and is defined as:

$$D_x = \partial_x + \frac{\theta}{x}(1 - R_x), \qquad (1)$$

where R_x denotes the reflection operator with respect to x, and θ is the Dunkl parameter, typically constrained by $\theta \geq -1/2$. The operator behaves like a standard derivative on even functions but exhibits distinct behavior on odd functions, reflecting its sensitivity to parity.

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Dunkl operators have facilitated the development of commutation relations in systems with reflection symmetry, finding applications in quantum mechanics and generalized Fourier analysis. Notably, they lead to a modified form of the Schrödinger equation, known as the Dunkl-Schrödinger equation; which enables a more precise description of wave functions in parity-sensitive physical systems such as crystals and interacting particles [1-5].

2. Thermodynamic geometry

An advanced approach to analyzing thermodynamic systems under generalized statistics is the application of geometric methods, known as *thermodynamic geometry* [6]. This concept was first introduced by Weinhold, who proposed a geometric formulation of thermodynamics by defining a metric based on the second derivatives of internal energy [7].

Different metrics can be defined depending on the chosen thermodynamic potential. In particular, the *Fisher-Rao metric* is formulated from the second derivatives of the logarithm of the partition function with respect to intensive parameters as follows [8,9]:

$$g_{ij} = \frac{\partial^2 \ln(z)}{\partial\beta^i \partial\beta^j}$$

Where β^i refers to the intensive thermodynamic parameters.The thermodynamic curvature is given by the Ricci scalar of the thermodynamic parameter space. It has been shown that the thermodynamic curvature of a classical ideal gas is zero, indicating the absence of intrinsic particle interactions. For ideal Bose (Fermi) gases, the curvature is positive (negative), reflecting intrinsic attractive (repulsive) statistical interactions. Moreover, singularities in the thermodynamic curvature are indicative of phase transitions [10].

3. Dunkl-Boson Distribution Function

In systems where bosonic oscillators are defined using Dunkl operators, the standard commutation relations are modified as:

$$\left[a_i^D, a_j^{\dagger D}\right] = \delta_{ij}(1 + 2\theta R_i)$$

with θ being the Dunkl parameter.This leads to a generalized occupation number distribution:

$$n_D(\epsilon) = \frac{2}{-1 + e^{2\beta\epsilon} z^{-2}} + \frac{1}{1 + e^{\beta\epsilon} e^{2\theta\beta\epsilon} z^{-1}}$$

where $z = e^{-\gamma} = e^{\mu/KT}$ is the fugacity. When θ=0, the standard Bose-Einstein distribution is recovered. This distribution reflects how Dunkl deformation affects quantum statistics, with implications for systems exhibiting reflection symmetry.

4. Thermodynamic quantities and Thermodynamic curvature

For the Dunkl-Boson distribution, internal energy and particle number for an ideal 3 dimensional gas become:

$$U = -\frac{3\,Li_{\frac{5}{2}}(z)}{2\left(\beta(1+2\theta)\right)^{\frac{5}{2}}} + \frac{3\,Li_{\frac{5}{2}}(z^2)}{4\sqrt{2}\,\beta^{\frac{5}{2}}}, \qquad N = -\frac{Li_{\frac{5}{2}}(z)}{\left(\beta(1+2\theta)\right)^{\frac{3}{2}}} + \frac{3\,Li_{\frac{3}{2}}(z^2)}{\sqrt{2}\,\beta^{\frac{3}{2}}}.$$

The thermodynamic metric components are:

$$g_{\beta\beta} = -\frac{3}{8}\left(\frac{10\,(1+2\theta)Li_{\frac{5}{2}}(-z)}{(\beta+2\beta\theta)^{\frac{7}{2}}} - \frac{5\,Li_{\frac{5}{2}}(z^2)}{\sqrt{2}\,\beta^{\frac{5}{2}}}\right),$$

$$, \quad g_{\beta\gamma} = g_{\gamma\beta} = -\frac{3}{8}\left(\frac{4\,(1+2\theta)Li_{\frac{3}{2}}(-z)}{(\beta+2\beta\theta)^{\frac{5}{2}}} - \frac{2\sqrt{2}\,Li_{\frac{3}{2}}(z^2)}{\sqrt{2}\,\beta^{\frac{5}{2}}}\right)$$

$$.g_{\gamma\gamma} = \left(\frac{(1+2\theta)Li_{\frac{1}{2}}(-z)}{(\beta+2\beta\theta)^{\frac{3}{2}}} - \frac{\sqrt{2}\,Li_{\frac{1}{2}}(z^2)}{\beta^{\frac{3}{2}}}\right)$$

Using these, the thermodynamic curvature can be computed [11,12]. Although its full expression is lengthy, numerical results are shown in Figure (1):

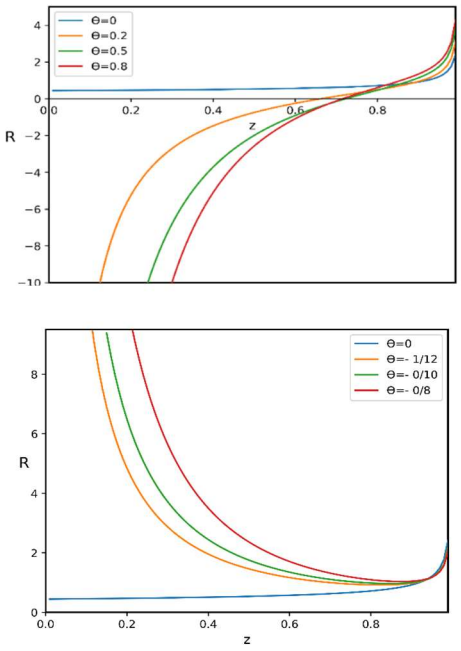


Fig. (1): Behavior of Thermodynamic Curvature vs. Fugacity for Different Dunkl Parameter Values (Isothermal Process, β = 1):

5. Conclusion

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In this study, the thermodynamic properties of Dunkl-Boson systems and the influence of the Dunkl parameter θ were investigated. Using generalized algebra and statistical relations, thermodynamic functions and distributions were derived, reducing to the standard Bose-Einstein case when θ=0. The thermodynamic curvature analysis revealed fermion-like (repulsive) behavior for positive θ\thetaθ at low fugacity and boson-like (attractive) behavior at higher values. A divergence at z=1 suggests a phase transition akin to Bose-Einstein condensation. These findings highlight the potential of Dunkl deformation in modeling crossover behaviors, possibly analogous to Cooper pair formation in fermionic systems.

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